Fractional Brownian motion and multifractional Brownian motion of Riemann-Liouville type

2001 J. Phys. A: Math. Gen. 34 1301

(http://iopscience.iop.org/0305-4470/34/7/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 1301–1310 www.iop.org/Journals/ja PII: S0305-4470(01)17728-0

Fractional Brownian motion and multifractional Brownian motion of Riemann–Liouville type

S C Lim

School of Applied Physics, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia

Received 2 October 2000, in final form 10 January 2001

Abstract

The relationship between standard fractional Brownian motion (FBM) and FBM based on the Riemann–Liouville fractional integral (or RL-FBM) is clarified. The absence of stationary property in the increment process of RL-FBM is compensated by a weaker property of local stationarity, and the stationary property for the increments of the large-time asymptotic RL-FBM. Generalization of RL-FBM to the RL-multifractional Brownian motion (RL-MBM) can be carried out by replacing the constant Hölder exponent by a time-dependent function. RL-MBM is shown to satisfy a weaker scaling property known as the local asymptotic self-similarity. This local scaling property can be translated into the small-scale behaviour of the associated scalogram by using the wavelet transform.

PACS numbers: 0540J, 230B, 0545D

1. Introduction

Fractional Brownian motion (FBM) is used widely in modelling phenomena with power spectra of the form $\omega^{-\gamma}$, $1 < \gamma < 3$. Examples for such applications include anomalous diffusion, biomedical modelling, earthquake modelling, financial time series analysis, etc. The standard FBM employed in most applications was introduced by Mandelbrot and Van Ness [1] using the following definition:

$$B_H(t) = \frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}[(t-u)^{H-1/2} - (-u)^{H-1/2}]\,\mathrm{d}B(u) + \int_0^t (t-u)^{H-1/2}\,\mathrm{d}B(u)\right\} \tag{1}$$

where $B(t)$ is the standard Brownian motion, Γ is the gamma function and the Hölder exponent (or Hurst index) H lies in the range $0 < H < 1$. Equation (1) is also known as the 'moving average' representation of FBM and can be written more compactly as [2]

$$B_H(t) = \frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{+\infty}[(t-u)_+^{H-1/2} - (-u)_+^{H-1/2}]\,\mathrm{d}B(u) \tag{2}$$

where $(x)_+ = \max(x, 0)$. Note that B_H is continuous, everywhere non-differentiable with a unique scaling exponent H for all t, which reflects the monofractal or homogeneous fractal character of the process. There also exists an equivalent representation of B_H (up to a multiplicative constant) known as the harmonizable or the spectral representation [3] which takes the form

$$B_H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\xi} - 1}{|\xi|^{H+1/2}} \, dB(\xi). \tag{3}$$

B_H is a Gaussian process with zero mean and its variance and covariance are respectively

$$\langle (B_H(t))^2 \rangle = \sigma_H^2 |t|^{2H} \tag{4}$$

$$\langle (B_H(t) B_H(s) \rangle = \frac{\sigma_H^2}{2} [|t|^{2H} + |s|^{2H} - |t - s|^{2H}] \tag{5}$$

with

$$\sigma_H^2 = \langle (B_H(1))^2 \rangle = \frac{\Gamma(1 - 2H) \cos(\pi H)}{\pi H}. \tag{6}$$

The standard FBM has the following 'desirable' properties. It is a self-similar process with the scaling exponent H:

$$B_H(at) = a^H B_H(t) \qquad \forall a > 0 \quad t \in \mathbb{R} \tag{7}$$

where the equality is in the sense of finite joint distributions. Though B_H is itself non-stationary, its increment process

$$\Delta B_H(t; \tau) \equiv B_H(t + \tau) - B_H(t) \qquad \tau > 0 \tag{8}$$

is stationary with covariance

$$\langle \Delta B_H(t; \tau_1) \Delta B_H(t; \tau_2) \rangle = \frac{\sigma_H^2}{2} [|\tau_1|^{2H} + |\tau_2|^{2H} - |\tau_1 - \tau_2|^{2H}]. \tag{9}$$

Self-similarity together with stationary increments imply

$$B_H(t + \tau) - B(t) = a^{-H} [B_H(t + a\tau) - B(t)] \qquad \forall a > 0 \quad t \in \mathbb{R}. \tag{10}$$

The stationary property of ΔB_H makes it possible to define a stationary fractional Gaussian noise as derivative of B_H (in the sense of distribution or generalized process), which allows a generalized power spectrum to be associated with B_H.

Though the standard FBM possesses some nice properties such as self-similarity and stationary increments, it does not represent a causal time-invariant system as there is no well-defined impulse response function. On the other hand, the 'one-sided' FBM introduced by Barnes and Allan [4] based on the following Riemann–Liouville (RL) fractional integral

$$V_H(t) = \frac{1}{\Gamma(H + 1/2)} \int_0^t (t - u)^{H-1/2} \, dB(u) \tag{11}$$

represents a linear system driven by white noise $\eta(t)$, with the impulse response function $t^{H-1/2}(\Gamma(H + 1/2))^{-1}$. (Note that the white noise is formally related to the Brownian motion by $dB(t) = \eta(t)\, dt$.) The RL-FBM $V_H(t)$ is a zero-mean Gaussian process with a rather complicated covariance:

$$\langle V_H(t) V_H(s) \rangle = \frac{t^{H-1/2} s^{H+1/2}}{(H + 1/2)(\Gamma(H + 1/2))^2} \, {}_2F_1(1, 1/2 - H, 3/2 + H, s/t) \tag{12}$$

where $s < t$ and ${}_2F_1$ is the Gauss hypergeometric function. However, the variance of V_H has the same time dependence as B_H:

$$\langle (V_H(t))^2 \rangle = \frac{t^{2H}}{2H(\Gamma(H + 1/2))^2}. \tag{13}$$

Despite its complex covariance, V_H shares with B_H many properties which include self-similarity, regularity of sample paths, etc—with one notable exception that its increment process is non-stationary. The absence of stationary property in its increments imply V_H cannot be associated with a generalized spectrum of power-law type as in the case of B_H. This is the main reason that RL-FBM is seldom used in modelling phenomena with $\omega^{-\gamma}$-type power spectrum.

The main objective of this paper is to show that RL-FBM does not totally lack the stationary property, as commonly suggested (see, for example, [5]). We shall show that under various conditions the increment process of V_H can be stationary. The second part of the paper deals with the generalization of RL-FBM to RL-multifractional Brownian motion (RL-MBM). The local properties of RL-MBM are studied.

2. Conditions of stationarity for increments of RL-FBM

The usual reason given for the absence of stationarity in the increments of RL-FBM is the over-emphasis on the time origin in V_H. The following is a heuristic way of looking at this. The standard FBM can be considered as the sum of two independent Gaussian processes:

$$B_H(t) = Z_H(t) + V_H(t) \tag{14}$$

with

$$Z_H(t) = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{0} [(t-u)^{H-1/2} - (-u)^{H-1/2}]\,\mathrm{d}B\,(u). \tag{15}$$

Z_H represents 'a history of infinite past' of B_H, whereas V_H is the 'finite memory' part. Thus B_H has a 'headstart' over V_H, which begins at time zero with no past. The addition of the infinite past to V_H results in a process with stationary increments. One thus expects the large-time asymptotic RL-FBM which has acquired 'infinite past' to have stationary increments. This turns out to be true as we shall show later on.

It is interesting to note that a similar situation exists in the Ornstein–Uhlenbeck (OU) process. The finite-starting time OU process Y with $Y(0) = 0$ is given by

$$Y(t) = \sigma \int_0^t \mathrm{e}^{-\Omega(t-u)}\,\mathrm{d}B\,(u) \tag{16}$$

with σ and Ω positive constants. This process is non-stationary. However, by adding a process representing infinite past $\sigma \int_{-\infty}^{0} \mathrm{e}^{-\Omega(t-u)}\,\mathrm{d}B\,(u)$ to Y results in a stationary OU process. As in the case of V_H, the large-time asymptotic process of Y is stationary. The kernels in the definitions for these two processes have different growth properties, that is, power growth (11) for FBM and exponential growth (16) for OU process. As a result, the stationary property for the two processes occurs at different levels, with the OU process being itself stationary, whereas the FBM acquires stationarity in its increments. We remark that the Markov property satisfied by the OU process is absent in FBM since non-exponential kernel implies memory. In particular, the power-law kernel of FBM represents a long memory process.

In order to see whether there exists some kind of stationary property in the increment property ΔV_H of the RL-FBM, it is sufficient to consider its variance since for a real process the covariance of its increment process can be obtained from the variance using the following identity:

$$\langle \Delta V_H(t,v)\Delta V_H(s,u)\rangle = \tfrac{1}{2}[\langle \Delta V_H(t;u)^2\rangle + \langle \Delta V_H(s;v)^2\rangle - \langle \Delta V_H(u;v)^2\rangle - \langle \Delta V_H(s;t)^2\rangle]. \tag{17}$$

First we note that the variance of ΔV_H satisfies the following estimates:

$$\langle(\Delta V_H(t;\tau))^2\rangle \leqslant \frac{1}{(\Gamma(H+1/2))^2}\int_{-\infty}^{+\infty}[(\tau+u)_+^{H-1/2}-u_+^{H-1/2}]^2\,\mathrm{d}u \leqslant C_H\tau^{2H} \tag{18}$$

where C_H is a positive constant. This inequality implies the sample paths of V_H are Hölder regular with exponent H.

The conditions for ΔV_H to be stationary can be obtained by considering the conditions for the variance of $\Delta V_H(t;\tau)$ to be independent of t. With some changes of variables one gets

$$\begin{aligned}\langle(\Delta V_H(t;\tau))^2\rangle &= \frac{1}{(\Gamma(H+1/2))^2}\left\{\int_0^t[(t+\tau-u)^{H-1/2}-(t-u)^{H-1/2}]^2\,\mathrm{d}u + \int_t^{t+\tau}(t+\tau-u)^{2H-1}\,\mathrm{d}u\right\}\\ &= \frac{\tau^{2H}}{(\Gamma(H+1/2))^2}\left\{\int_0^{t/\tau}[(1+u)^{H-1/2}-u^{H-1/2}]^2\,\mathrm{d}u+\int_0^t u^{2H-1}\,\mathrm{d}u\right\}\\ &= \frac{\tau^{2H}}{(\Gamma(H+1/2))^2}\left\{I+\frac{1}{2H}\right\}.\end{aligned} \tag{19}$$

The integral I in (14) is independent of t provided (a) $t/\tau \to 0$, or (b) $t/\tau \to \infty$.

Condition (a) gives $I = 0$. However this condition is satisfied for very large time lag τ. Such a condition is too restrictive for applications. When condition (b) is satisfied, one gets $[\Gamma(H+1/2)]^2 I = \Gamma(1-2H)\cos(\pi H)/(\pi H) = \sigma_H^2$. There are two possible ways to fulfil condition (b): either $t \to \infty$ for all τ, or $\tau \to 0$ for all t. The requirement that $t \to \infty$ means that the increment process of the large-time asymptotic RL-FBM is stationary. This result has been obtained previously using a rather cumbersome approximation with identities of Gauss hypergeometric functions and lengthy algebra [6]. On the other hand, the condition $\tau \to 0$ implies the increment process of RL-FBM is locally asymptotically stationary with

$$\langle(\Delta V_H(t;\tau))^2\rangle = D_H\tau^{2H} \qquad \tau \to 0 \tag{20}$$

where $D_H = \sigma_H^2 + [2H(\Gamma(H+1/2))^2]^{-1}$.

For most practical applications, both the conditions for stationary increments (i.e. $t \to \infty$ and $\tau \to 0$) are rather too stringent. It would be useful if the conditions could be suitably relaxed. This can be achieved by assuming τ small enough such that terms of $\vartheta(\tau^2)$ and higher powers can be neglected. A change in variables and evaluating the integral I up to order $\vartheta(\tau^2)$ gives

$$\begin{aligned}I &= t^{2H}\int_0^1\left[\left(u+\frac{\tau}{t}\right)^{H-1/2}-u^{H-1/2}\right]^2\mathrm{d}u\\ &\approx (H-1/2)^2 t^{2(H-1)}\tau^2\int_0^1 u^{2H-3}\,\mathrm{d}u\\ &= \frac{(H-1)^2}{8(H+1)}t^{2(H-1)}\tau^2.\end{aligned} \tag{21}$$

Thus the increment process of V_H is stationary for τ sufficiently small so that $\vartheta(\tau^2) \approx 0$. Since $(H-1) < 0$, $t^{2(H-1)}$ decreases as t increases. Therefore τ can take larger values progressively as t increases such that $t^{2(H-1)}\tau^2 \approx 0$ still holds. In other words, the size of the interval of stationarity for the increment process $\Delta V_H(t;\tau)$ is time dependent. Between the two extreme stationarity conditions $t \to \infty$ and $\tau \to 0$ there exist 'intermediate' conditions which allow the interplay of the values of t and τ. The interval of approximate stationarity for the increments of RL-FBM is very much smaller at the beginning than at large t. We call this latter property the local stationary property since it is the consequence of the local assumption $\tau \ll t$. Local stationary increments provide some flexibility necessary for practical applications.

We can now look more closely the relationship between the standard FBM B_H and the RL-FBM V_H. The statement that $V_H(t) \to B_H(t)$ as $t \to \infty$ which has been taken for granted can be checked by calculating the covariance of V_H for $s, t \to \infty$:

$$\langle V_H(s)V_H(t)\rangle = \tfrac{1}{2}\{\langle (V_H(s))^2\rangle + \langle (V_H(t))^2\rangle - \langle (V_H(t) - V_H(s))^2\rangle\}$$
$$= \frac{1}{2H(\Gamma(H+1/2))^2}[|s|^{2H} + |t|^{2H} - 2H(\Gamma(H+1/2))^2 D_H|t-s|^{2H}] \qquad (22)$$

which differs from the covariance of B_H in the coefficient of $|t-s|^{2H}$ term. Despite this minor difference, the large-time asymptotic RL-FBM satisfies all the properties of B_H. On the other hand, the increment process of RL-FBM $\Delta V_H(t;\tau)$ approaches (up to a multiplicative constant) the increment process of standard FBM $\Delta B_H(t;\tau)$ in both the limits $\tau \to 0$ and $t \to \infty$. By using (21) and omitting the $\vartheta(\tau^2)$ term, one notes that locally V_H approaches B_H since the local covariances of V_H and B_H have the same time dependence:

$$\langle B_H(t)B_H(t+\tau)\rangle \sim \langle V_H(t)V_H(t+\tau)\rangle \sim |t|^{2H} + |t+\tau|^{2H} - |\tau|^{2H} \qquad \tau \ll t. \qquad (23)$$

From the above discussion, one sees that the increment process of RL-FBM is not totally lacking of stationary property. Instead, it satisfies some weaker forms of stationarity.

3. Riemann–Liouville multifractional Brownian motion

FBM can only be used in modelling phenomena which have the same irregularities globally or monofractal structure because it has a constant Hölder exponent. In order to consider phenomena which have more intricate structures with variations in irregularities, it is necessary to allow the Hölder exponent to vary as a function of time (or position). A direct way of extending the monofractal FBM to a multifractal FBM or MBM is to replace the constant Hölder exponent by $H(t)$, $t \in \mathbb{R}_+$, a (0,1)-valued function with Hölder regularity r, $r > \sup H(t)$. This time-varying Hölder exponent $H(t)$ describes the local variations of the irregularity of the MBM process. Note that in general $H(t)$ can be a deterministic or random function, and it needs not be a continuous function. In this paper, however, we shall restrict $H(t)$ to be a smooth deterministic function of time.

Generalization of the standard FBM B_H to the standard MBM $B_{H(t)}$ was first carried out independently by Peltier and Lévy-Véhel [7] based on the moving-average representation of FBM and by Benassi *et al* [8] using the spectral representation of FBM. These two generalizations of MBM have been shown to be almost certainly equivalent up to a multiplicative deterministic function of time [9, 10].

$B_{H(t)}$ locally behaves very much like FBM B_H. Due to the time dependence of the Hölder exponent, MBM does not satisfy the self-similar property globally, and its increment process is no longer stationary. However, if an additional assumption is imposed on $H(t)$ such that $H(t) \in C^r(\mathbb{R}, (0,1))$, $t \in \mathbb{R}$ for some positive r with $r > \sup H(t)$, then it can be shown that $H(t_0)$ is almost certainly the Hölder exponent of the MBM at the point t_0; and the local Hausdorff and box dimensions of the graph of $B_{H(t)}$ at t_0 are almost certainly $2 - H(t_0)$. Since the scaling exponent is time dependent, MBM fails to satisfy the global self-similar property. One may consider a local scaling property by replacing H in (7) by $H(t)$ such that $B_{H(t)}(at) = a^{H(t)}B_{H(t)}(t)$ for all $a > 0$. However, such a definition of local self-similarity can lead to the invalid consequence that the law of the process at all times is dependent on the $H(t)$ at a particular time. A more satisfactory characterization of local self-similarity is as follows [8]. A process $X(t)$ is locally asymptotically self-similar at point t_0 if

$$\lim_{\rho\to 0^+}\left[\frac{X(t_0+\rho u)-X(t_0)}{\rho^{H(t_0)}}\right]_{u\in\mathbb{R}} = (B_{H(t_0)}(u))_{u\in\mathbb{R}} \tag{24}$$

where the equality is up to a multiplicative deterministic function of time. The standard MBM is locally asymptotically self-similar in the above sense. Thus MBM at a time t_0 behaves locally like a FBM with Hölder exponent $H(t_0)$.

Now we want to consider another kind of MBM which is the extension of the RL-FBM V_H to the RL-MBM $V_{H(t)}$. If the Hölder exponent in the definition of RL-FBM (10) is replaced by $H(t)$ which satisfies all the conditions stated above, the resulting RL-MBM is a Gaussian process with mean zero and the following covariance:

$$\langle V_{H(s)}(s)V_{H(t)}(t)\rangle = \frac{{}_2F_1\,(1, 1/2-H(t), H(s)+3/2, s/t)}{(2H(s)+1)\Gamma(H(s)+1/2)\Gamma(H(t)+1/2)} s^{H(s)+1/2}t^{H(t)-1/2} \tag{25}$$

for $t > s > 0$. RL-MBM was first introduced by Lim and Muniandy [10] who have shown that for $t \gg 1$, $V_{H(t)}$ is locally asymptotically self-similar. Here we shall generalize this result and give a simple proof to show that the RL-MBM is locally asymptotically self-similar for all t, just like the standard MBM.

First one notes that for $\tau \to 0^+$, $H(t+\tau) \approx H(t)$ due to the assumption that $H(t)$ is continuous. Now consider the increment process of RL-MBM, $\Delta V_{H(t)}(t;\tau) \equiv V_{H(t)}(t+\tau) - V_{H(t)}(t)$ for $\tau \to 0^+$, its variance can be computed in a similar way as in the case for RL-FBM, with all H replaced by $H(t)$:

$$\langle(\Delta V_{H(t)}(t;\tau))^2\rangle = D_{H(t)}(t)\tau^{2H(t)} \qquad \tau\to 0^+ \tag{26}$$

where

$$D_{H(t)}(t) = \frac{\Gamma(1-2H(t))\cos(\pi H(t))}{\pi H(t)} - \frac{1}{2H(t)(\Gamma(H(t)+1/2))^2}. \tag{27}$$

In the time interval $[t+\tau, t]$, $\tau\to 0^+$, $D_{H(t)}$ can be taken as a constant so that the increment process of RL-MBM is said to be locally asymptotically stationary. Unlike the RL-FBM, this is the only kind of stationary property satisfied by increment process of RL-MBM for all $t > 0$. The local covariance of the RL-MBM can be calculated using (25) and $H(t+\tau) \approx H(t)$ for $\tau \to 0^+$:

$$\begin{aligned}\langle V_{H(t)}(t)V_{H(t+\tau)}(t+\tau)\rangle = {} & \frac{1}{2H(t)(\Gamma(H(t)+1/2))^2}\,[|t|^{2H(t)} + |t+\tau|^{2H(t)} \\ & -2H(t)(\Gamma(H(t)+1/2))^2 D_{H(t)}|\tau|^{2H(t)}].\end{aligned} \tag{28}$$

Finally, we show that the RL-MBM satisfies the locally asymptotically self-similar property just like the standard MBM. For $u, v \in \mathbb{R}_+$ and $\rho \to 0^+$, one gets by using (26) that

$$\left\langle \frac{V_{H(t_0)}(t_0+\rho u)V_{H(t_0)}(t_0+\rho v)}{\rho^{2H(t_0)}}\right\rangle = D_{H(t_0)}[|u|^{2H(t_0)} + |v|^{2H(t_0)} - |v-u|^{2H(t_0)}] \tag{29}$$

which verifies the locally asymptotically self-similar condition:

$$\lim_{\rho\to 0^+}\left[\frac{V_{H(t_0)}(t_0+\rho u) - V_{H(t_0)}(t_0)}{\rho^{H(t_0)}}\right]_{u\in\mathbb{R}_+} = (B_{H(t_0)}(u))_{u\in\mathbb{R}_+}. \tag{30}$$

Again the equality is up to a multiplicative deterministic function of time. This property implies the existence of a tangent FBM $B_{H(t_0)}$ at each time t_0 where the RL-MBM is defined. The Hölder exponent of this local FBM is given by $H(t_0)$.

For application purposes, the following characterization of the local behaviour of $V_{H(t)}$ may be useful. Let $H_\varepsilon(t) = H(t/\varepsilon)$, $D_{H_\varepsilon(t)} = D_{H(t/\varepsilon)}$ and

$$V_{H_\varepsilon(t)}(t) = \frac{1}{\Gamma(H_\varepsilon(t)+1/2)} \int_0^t (t-u)^{H_\varepsilon(t)-1/2}\, \mathrm{d}B\,(u). \tag{31}$$

The variance of the increment process of $V_{H_\varepsilon(t)}$ is

$$\langle (V_{H_\varepsilon(t)}(t+\tau) - V_{H_\varepsilon(t)}(t))^2 \rangle \approx D_{H_\varepsilon(t)} |\tau|^{2H_\varepsilon(t)} \tag{32}$$

for ε sufficiently small such that the increment process is stationary over $\tau \ll \varepsilon^{-1}$. The parameter ε can be regarded as a measure of the interval of stationarity as it specifies the size of neighbourhood of t for which the increment process of $V_{H_\varepsilon(t)}$ is approximately stationary. For scales which are smaller than the interval of stationarity the process $V_{H_\varepsilon(t)}(t)$ behaves locally like a FBM with Hölder exponent frozen at t.

4. Time-scale analysis of RL-FBM and RL-MBM

Since both non-stationary features and scaling (global as well as local) properties are simultaneously involved in RL-FBM and RL-MBM, time-scale analysis based on wavelet transforms seems naturally relevant to these processes. The wavelet transform of a function (or process) $X(t)$, $TX(t,a)$, can be regarded as a kind of mathematical microscope at temporal position t with the length scale a at which X is examined. The small-scale properties of the wavelet transform make it a very useful tool for characterizing the local regularity of a process.

Let ψ be the mother wavelet with compact support and satisfying the usual admissibility condition of vanishing moment:

$$\int_{-\infty}^{+\infty} \psi(t)\,\mathrm{d}t = 0. \tag{33}$$

Recall that for the standard FBM, the stationarity of its increments together with the vanishing moment condition result in a stationary wavelet transform process $TB_H(t,a)$ [11, 12]. Since the increment process of the RL process is locally stationary, one needs to apply the time-scale analysis to V_H locally in order to obtain an analogous result as for B_H. The wavelet transform of V_H with respect to wavelet ψ is

$$TV_H(t,a) \equiv T_V(t,a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} V_H(t')\psi\left(\frac{t-t'}{a}\right)\mathrm{d}t' \tag{34}$$

for a fixed scale $a > 0$. The covariance of the wavelet process is given by

$$\begin{aligned}\langle T_V(s,a)T_V(t,b)\rangle &= \frac{1}{\sqrt{ab}} \iint \psi\left(\frac{s-s'}{a}\right)\psi\left(\frac{t-t'}{b}\right)\langle V_H(s')V_H(t')\rangle\,\mathrm{d}s'\,\mathrm{d}t' \\ &= \sqrt{ab}\iint \psi(u)\psi(v)\langle V_H(s-au)V_H(t-bv)\rangle\,\mathrm{d}u\,\mathrm{d}v. \end{aligned}\tag{35}$$

In order to obtain the scalogram one lets $s = t$, $a = b \to 0^+$ and together with the local covariance of V_H and the vanishing moment condition

$$\begin{aligned} S(t,a) &= \langle (T_V(t,a))^2\rangle \\ &= a\iint \psi(u)\psi(v)\langle V_H(t-au)V_H(t-av)\rangle\,\mathrm{d}u\,\mathrm{d}v \\ &= a^{2H+1}\Gamma_\psi \qquad a \to 0^+ \end{aligned}\tag{36}$$

where

$$\Gamma_\psi = -D_H \iint \psi(u)\psi(v)|u-v|^{2H}\,\mathrm{d}u\,\mathrm{d}v. \tag{37}$$

This shows that the wavelet transform of the RL-FBM gives a wavelet process which is locally asymptotically stationary. Note that despite the RL-FBM being self-similar for all t, the wavelet process does not satisfy the scaling property globally. This is due to the fact that the covariance used in (36) is a local one, which is derived based on the local stationary property of the increment process.

The application of the time-scale analysis to RL-MBM is similar to that for the RL-FBM. Since $H(t)$ is assumed to be a continuous function, $H(t+au) \approx H(t+av) \approx H(t)$ for $a \to 0^+$. The rest of the argument is the same as given above, with all H replaced by $H(t)$. Again, the result shows that the locally asymptotically self-similar property of RL-MBM is translated into the locally stationary behaviour of the wavelet process (that is for small scaling a).

A possible way of estimating H based on the scalogram is to use the following equation:

$$\log S = (2H+1)\log a + \log \Gamma_\psi. \tag{38}$$

A log–log plot of the scalogram versus the scale enables one to extract the value of H (or $H(t)$). This is just the basic idea of estimating the Hölder exponents. In practice this is a very difficult problem, since it is necessary to estimate the Hölder exponents over intervals that are short enough such that they can regarded as constant, but long enough so that their statistical estimates are stable. A more detailed discussion on this topic is given in [13].

5. Time–frequency analysis of RL-FBM and RL-MBM

From the stationary properties of the increment process of the RL-FBM one expects the process to exhibit similar power-law type of spectrum as the standard FBM for large time as well as over local intervals of stationary increments. The time–frequency analysis of the RL-FBM based on the Wigner–Ville distribution has been considered in [14]. It was found that the large-time average of the asymptotic Wigner–Ville distribution varies as $|\omega|^{(2H+1)}$.

We shall attempt to obtain a local 'spectrum' for the RL-FBM and MBM. For notational convenience, we let the increment process of RL-FBM over a sufficiently small time lag δ be denoted by

$$\Delta^\delta(t) = V_H(t) - V_H(t-\delta) \qquad \delta \ll t. \tag{39}$$

Define

$$\Delta^\delta_\varepsilon(t) = \Delta^\delta\left(\frac{t}{\varepsilon}\right) \tag{40}$$

where ε is sufficiently small such that the process is approximately stationary over intervals that are small compared to ε^{-1}, which is large. Let

$$u = \frac{t+s}{2} \qquad v = \frac{t-s}{2}.$$

The covariance of $\Delta^\delta_\varepsilon$ can be expressed as

$$\begin{aligned} R^\delta_\varepsilon(t,s) &= \langle \Delta^\delta_\varepsilon(t)\Delta^\delta_\varepsilon(s)\rangle \\ &= C^\delta_\varepsilon(u,v) = R^\delta_\varepsilon\left(u - \frac{\varepsilon v}{2}, u + \frac{\varepsilon v}{2}\right) \\ &\approx R^\delta_\varepsilon(v) \\ &= \frac{\sigma_H}{2}\left[\left|\frac{v+\delta}{\varepsilon}\right|^{2H} + \left|\frac{v-\delta}{\varepsilon}\right|^{2H} - 2\left|\frac{\delta}{\varepsilon}\right|^{2H}\right] \end{aligned} \tag{41}$$

as $\varepsilon \to 0$. By noting that the Fourier transform of the generalized function $f(\delta) = |\delta|^{2H}$ is $\hat{f}(\omega) = K_H|\omega|^{-2(H+1)}$ with $K_H > 0$, one can then approximate the time varying 'spectrum' (or Wigner–Ville distribution) as

$$\begin{aligned}\Omega_\varepsilon^\delta &= \int_{-\infty}^{\infty} C_\varepsilon^\delta(u, v)\mathrm{e}^{\mathrm{i}wv}\,\mathrm{d}v \\ &\approx \int_{-\infty}^{\infty} R_\varepsilon^\delta(v)\mathrm{e}^{\mathrm{i}wv}\,\mathrm{d}v \\ &= \frac{2\sigma_H^2 K_H}{\varepsilon^{2H}} \frac{\sin^2(\frac{\delta\omega}{2})}{|\omega|^{2H+1}} \qquad \delta, \varepsilon \ll 1.\end{aligned} \tag{42}$$

The above argument can also be applied to RL-MBM, provided the increment process $\Delta_\varepsilon^\delta(t)$ is to be replaced by

$$\Delta_\varepsilon^\delta(t) = V_{H(t/\varepsilon)}(t/\varepsilon) - V_{H(t/\varepsilon)}((t-\delta)/\varepsilon). \tag{43}$$

Here we have assumed $H((t+\delta)/\varepsilon) \approx H(t/\varepsilon)$ for $\delta \ll t$, $\varepsilon \ll t$. The local 'spectrum' for the RL-MBM is again given by (42) with H replaced by $H(t/\varepsilon)$.

6. Conclusion

We have shown that, contrary to the common belief that RL-FBM lacks any kind of stationary property, increments of the large-time asymptotic process are stationary, and the increments also satisfy a weaker stationary property—locally asymptotically stationary. The latter property extends naturally to MBM of RL-type.

The situation is quite different when both the standard FBM and RL-FBM are generalized to their respective MBM. Since both types of MBM do not have stationary increments, so the main 'advantage' that the standard FBM has over the RL-FBM does not hold any more. Now the two types of MBM have very similar local properties: in particular, when they are localized around t_0 by a positive scaling factor, then they asymptotically converge in distributions to a FBM with index $H(t_0)$ when the scaling factor goes to zero (that is they satisfy the locally asymptotically self-similar property). As regards applications in modelling and simulation, RL-MBM may turn out to be a better candidate than the standard MBM, in particular for processes which have finite starting time (see [13] for examples of such applications).

Acknowledgments

This research is supported by the Ministry of Science, Technology and Environment under IRPA grant no 09-02-02-0092. I would like to thank Universiti Kebangsaan Malaysia for the sabbatical leave, the Physics Department of the National University of Singapore where I spent part of my leave, Deutscher Akademischer Austauschdienst (DAAD) for an award which supported my research visit to Institut für Mathematik, Technische Universität Clausthal and my host Michael Demuth for his hospitality. I would also like to thank S V Muniandy for discussions.

References

[1] Mandelbrot B B and van Ness J W 1968 *SIAM Rev.* **10** 422
[2] Samorodnitsky G and Taqqu M S 1994 *Stable Non-Gaussian Random Processes* (New York: Chapman and Hall)
[3] Yaglom A M 1986 *Correlation Theory of Stationary and Related Random Functions* (New York: Springer)

[4] Allen D W and Barnes J A 1966 *Proc. IEEE* **54** 176
[5] Wornell G W 1996 *Signal Processing with Fractals: A Wavelet Approach* (Englewood Cliffs, NJ: Prentice-Hall) p 37
[6] Lim S C and Muniandy S V 1995 *Phys. Lett.* A **206** 311
[7] Peltier R F and Levy-Vehel J 1995 *INRIA Report* No 2645
[8] Benassi A, Jaffard S and Roux D 1997 *Rev. Math. Iber.* **13** 19
[9] Cohen S 1999 *Fractals: Theory and Applications in Engineering* ed M Dekking *et al* (New York: Springer) p 3
[10] Lim S C and Muniandy S V 2000 *Phys. Lett.* A **266** 140
[11] Flandrin P 1999 *Time-Frequency/Time-Scale Analysis* (San Diego: Academic)
[12] Mallat S 1998 *A Wavelet Tour of Signal Processing* (San Diego: Academic)
[13] Muniandy V M and Lim S C 2001 *Phys. Rev.* E at press
[14] Sithi V M and Lim S C 2000 *J. Phys. A: Math. Gen.* **28** 2995